Chapter 11: Special Theory of Relativity

Einstein's special theory of relativity is based on two postulates:

1. Laws of physics are invariant in form in all Lorentz frames (In relativity, we often call the inertial frame a Lorentz frame.)

2. The speed of light in vacuum has the same value *c* in all Lorentz frames, independent of the motion of the source.

The basics of the theory are covered in Appendix A on an elementary level with an emphasis on the Lorentz transformation and relativistic momentum/energy. Here, we examine relativity in the four-dimensional space of $\mathbf{x}$ and t, which provides the framework for us to examine the laws of mechanics and electromagnetism. The contents of the lecture notes now depart slightly from Ch.11 of Jackson.

Most of the materials presented here are consistent with Griffiths.

Griffiths

The Two Postulates

The two postulates in the theory of special relativity are:

1. **The principle of relativity**: All physical laws have the same form in all inertial frames.
2. **The universal speed of light**: The speed of light in free space is the same in all inertial frames. It does not depend on the motion of the source or the observer.

Both postulates are restricted to inertial frames. This is why the theory is special.

•The principle of relativity extends the concept of covariance from mechanics to all physical laws.

•The constancy of the speed of light is difficult to accept at first.

All the experimental consequences have confirmed its correctness.

Section 1: Definitions and Operation Rules of Tensors of Different Ranks in the 4-Dimensional Space

The Lorentz Transformation :

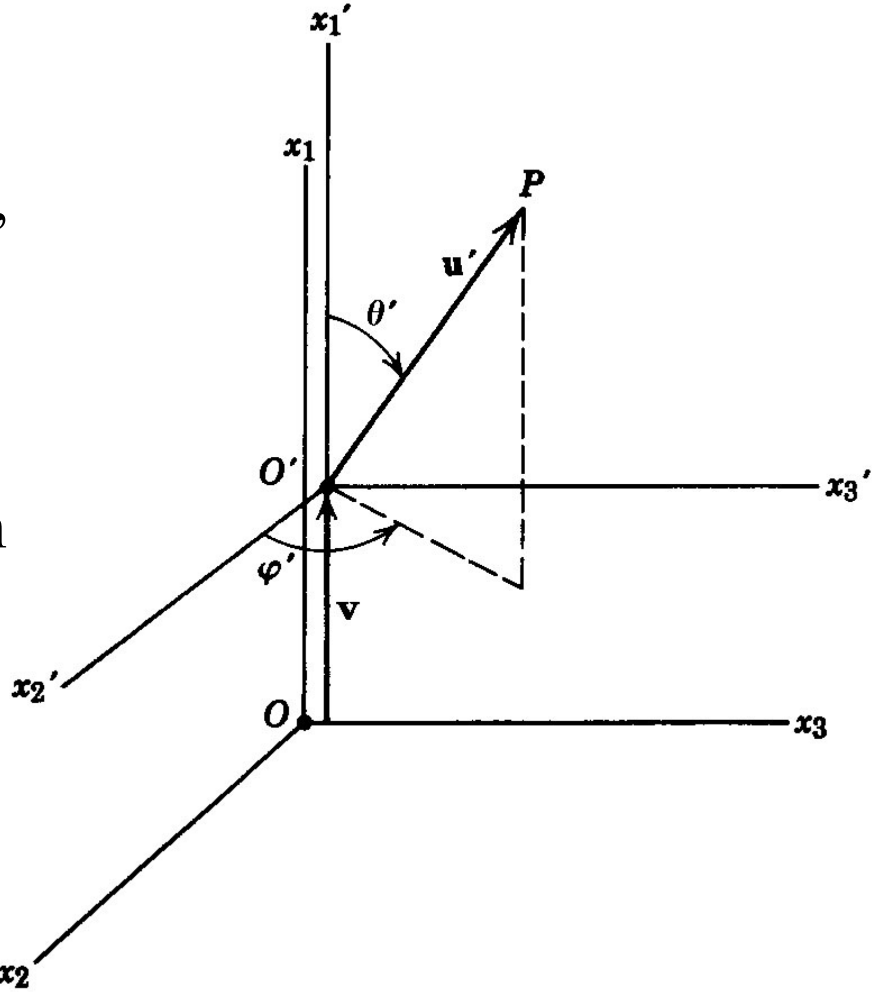
Consider two Lorentz frames, K and K'. Frame K' moves along the common x-axis with constant speed v relative to frame K.

Assume that at $t = t' = 0$, coordinate axes of frames K and K' overlap. Postulate 2 leads to the following Lorentz transformation for space and time coordinates. [derived in Appendix A, Eq. (A.15), where the relative motion is assumed to be along the x-axis.]

$$\begin{cases} x_0' = \gamma(x_0 - \beta x_1) \\ x_1' = \gamma(x_1 - \beta x_0) \\ x_2' = x_2 \\ x_3' = x_3 \end{cases} \quad (11.16) \qquad \begin{cases} x_0 = \gamma(x_0' + \beta x_1') \\ x_1 = \gamma(x_1' + \beta x_0') \\ x_2 = x_2' \\ x_3 = x_3' \end{cases} \quad (11.18)$$

where $\beta = v / c$ and $\gamma = (1 - \beta^2)^{-1/2}$ (11.17, the Lorentz factor).

A note about notation: In many books, the relative speed between two frames is denoted by v and the particle velocity in a given frame is denoted by **u**. This eventually leads to two definitions for the same notation γ:



Lorentz factor for the transformation, Jackson (11.17)

$$\gamma \equiv \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

Lorentz factor of a particle in a given frame, Jackson (11.46) and (11.51)

$$\gamma \equiv \left(1-\frac{u^2}{c^2}\right)^{-\frac{1}{2}}$$

Lorentz Transformation

The space-time continuum is defined in terms of a four-dimensional space with coordinates $x^0,\ x^1, x^2,\ x^3$.

Suppose that there is a well-defined transformation that yields new coordinates $x'^0,\ x'^1, x'^2,\ x'^3$, according to some rule,

$$x'^\alpha = x'^\alpha(x^0,\ x^1, x^2,\ x^3) \qquad (\alpha = 0,\ 1,\ 2,\ 3) \qquad (11.60)$$

$$dx'^\alpha = \frac{\partial x'^\alpha}{\partial x^0}dx^0 + \frac{\partial x'^\alpha}{\partial x^1}dx^1 + \frac{\partial x'^\alpha}{\partial x^2}dx^2 + \frac{\partial x'^\alpha}{\partial x^3}dx^3$$

μ	mu
ν	nu
v	vi
υ	*Italic*

$$\begin{pmatrix} dx'^0 \\ dx'^1 \\ dx'^2 \\ dx'^3 \end{pmatrix} = \begin{pmatrix} \frac{\partial x'^0}{\partial x^0} & \frac{\partial x'^0}{\partial x^1} & \frac{\partial x'^0}{\partial x^2} & \frac{\partial x'^0}{\partial x^3} \\ \frac{\partial x'^1}{\partial x^0} & \frac{\partial x'^1}{\partial x^1} & \frac{\partial x'^1}{\partial x^2} & \frac{\partial x'^1}{\partial x^3} \\ \frac{\partial x'^2}{\partial x^0} & \frac{\partial x'^2}{\partial x^1} & \frac{\partial x'^2}{\partial x^2} & \frac{\partial x'^2}{\partial x^3} \\ \frac{\partial x'^3}{\partial x^0} & \frac{\partial x'^3}{\partial x^1} & \frac{\partial x'^3}{\partial x^2} & \frac{\partial x'^3}{\partial x^3} \end{pmatrix} \begin{pmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{pmatrix}$$

$$dx'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu dx^\nu$$

$$\Lambda^\mu_\nu = \frac{\partial x'^\mu}{\partial x^\nu}$$

For the moment the transformation law is not specified.

Lorentz Transformation Matrix

$$x^0 \equiv ct,\ x^1 = x,\ x^2 = y,\ x^3 = z,\ \text{and } \beta = v/c$$

$$\left.\begin{aligned} x'^0 &= \gamma(x^0 - \beta x^1), \\ x'^1 &= \gamma(x^1 - \beta x^0) \\ x'^2 &= x^2 \\ x'^3 &= x^3 \end{aligned}\right\} \text{the Lorentz transformations}$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{the Lorentz transformation matrix}} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu x^\nu$$

$$dx'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu dx^\nu$$

$$\Lambda^\mu_\nu = \frac{\partial x'^\mu}{\partial x^\nu}$$

Covariant and Contravariant Vectors

Any set of 4 quantities A^α $(\alpha = 0 - 3)$ or $\mathbf{A} = (A^0, A^1, A^2, A^3)$, which transform in the same way as x^α, i.e.,

$$A'^\alpha = \sum_\beta \Lambda^\alpha_\beta A^\beta = \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta, \qquad (11.61)$$

is called a 4-vector. ↑ The Einstein summation convention

For tensors of rank one, called vectors, two kinds must be distinguished.

Contravariant vector: $A'^\alpha = \frac{\partial x'^\alpha}{\partial x^0} A^0 + \frac{\partial x'^\alpha}{\partial x^1} A^1 + \frac{\partial x'^\alpha}{\partial x^2} A^2 + \frac{\partial x'^\alpha}{\partial x^3} A^3$

Covariant vector: $B'_\alpha = \frac{\partial x^0}{\partial x'^\alpha} B_0 + \frac{\partial x^1}{\partial x'^\alpha} B_1 + \frac{\partial x^2}{\partial x'^\alpha} B_2 + \frac{\partial x^3}{\partial x'^\alpha} B_3$

$$B'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha} B_\beta, \qquad (11.62)$$

Metric/Norm Tensor (I)

What is the difference between x^α and x_α ? $\Rightarrow$ Metric/norm tensor.

In special theory of relativity, Lorentz transformation of the four-dimensional coordinates follow from the invariance of two events:

$$(ds)^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \tag{11.67}$$

$$(ds)^2 = g_{\alpha\beta} dx^\alpha dx^\beta \tag{11.68}$$

$g_{\alpha\beta} = g_{\beta\alpha}$ is called the *metric tensor* and diagonal in *flat* space-time.

$$g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -1 \tag{11.69}$$

The contravariant metric tensor $g^{\alpha\beta}$ is defined as the normalized co-factor of $g_{\alpha\beta}$. For flat spac-time it is the same:

$$g^{\alpha\beta} = g_{\alpha\beta} \tag{11.70}$$

Metric/Norm Tensor (II)

The contraction of the contravariant and covariant metric tensors give the Kronecker delta in four dimensions:

$$g_{\alpha\gamma} g^{\gamma\beta} = \delta_\alpha^\beta \qquad (11.71)$$

where $\delta_\alpha^\beta = 0$ for $\alpha \neq \beta$ and $\delta_\alpha^\alpha = 1$ for $\alpha = 0, 1, 2, 3.$

$$x_\alpha = g_{\alpha\beta} x^\beta \qquad (11.72)$$

$$x^\alpha = g^{\alpha\beta} x_\beta \qquad (11.73)$$

With the metric tensor, a contravariant vector and a co-variant vector can be expressed as:

$$A^\alpha = (A^0, \mathbf{A}), \quad \text{and} \quad A_\alpha = (A^0, -\mathbf{A}) \qquad (11.75)$$

Length contraction

The transformations between two inertial systems S and S' are $x' = \gamma_v(x - vt)$ and $t' = \gamma_v(t - vx/c^2)$. Show that when $\Delta t = 0$, $\Delta x = \Delta x'/\gamma_v$; but when $\Delta t' = 0$, $\Delta x' = \Delta x/\gamma_v$.

Explain why the length relations depend on simultaneity.

1. $\Delta t = 0 \Rightarrow \Delta x' = \gamma_v \Delta x \quad \Rightarrow \Delta x = \dfrac{1}{\gamma_v}\Delta x'$ ($\Delta x'$: the proper length)

2. $\Delta t' = 0 \Rightarrow \Delta t = \dfrac{v}{c^2}\Delta x \quad \Rightarrow \Delta x' = \gamma_v(\Delta x - v\Delta t) = \gamma_v(\Delta x - \dfrac{v^2}{c^2}\Delta x) = \dfrac{1}{\gamma_v}\Delta x$

   $\Rightarrow \Delta x' = \dfrac{1}{\gamma_v}\Delta x$ (Δx : the proper length)

Two events that are simultaneous in one inertial system are not simultaneous in another.

Question: How about the time dilation?

Velocity addition rule

The transformations between two inertial systems S and S' are $x' = \gamma_v(x - vt)$ and $t' = \gamma_v(t - vx/c^2)$. Suppose a particle moves a distance dx in S in a time dt. Its velocity $u = dx/dt$.
Show the velocity dx'/dt' in S'.

Sol: $dx' = \gamma_v(dx - vdt)$ and $dt' = \gamma_v(dt - vdx/c^2)$

$$u' = \frac{dx'}{dt'} = \frac{\gamma_v(dx - vdt)}{\gamma_v(dt - v\frac{dx}{c^2})} = \frac{(u - v)}{(1 - \frac{uv}{c^2})}$$

Special case: $u = c$ and $v = -c \Rightarrow u' = \dfrac{(c + c)}{(1 + c^2/c^2)} = c$

The speed of light in vacuum has the same value c in all Lorentz frames, independent of the motion of the source.

11.1 Definitions and Operation Rules of … (*continued*)

Four - Dimension Space Quantities and Operation Rules :

Define a position vector in the 4-dimensional space of **x** and *t* as

$$\mathbf{x} \equiv (ct,\ x,\ y,\ z) = (ct,\ \mathbf{x})$$

$\mu = 0-3,$ row number

and a 4-D matrix as $\Lambda^{\mu}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$

$\nu = 0-3,$ column number

$$\beta = v/c$$

$$\gamma = (1-\beta^2)^{-1/2}$$

then, the Lorentz transformation in (1) can be written

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$x'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu} x^{\nu}$$

$$\Lambda^{\mu}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}$$

The inner or scalar product is defined as the product of components of a covariant and a contravariant vector

$$B \cdot A \equiv B_\mu A^\mu \tag{11.66}$$

$$B' \cdot A' = \frac{\partial x^\beta}{\partial x'^\mu} B_\beta \frac{\partial x'^\mu}{\partial x^\lambda} A^\lambda = \frac{\partial x^\beta}{\partial x^\lambda} B_\beta A^\lambda = \delta_\lambda^\beta B_\beta A^\lambda = B \cdot A \tag{11.66$'$}$$

$$\frac{\partial x^\beta}{\partial x'^\mu} \frac{\partial x'^\mu}{\partial x^\lambda} = \delta_\lambda^\beta$$

Just as the 3-dimensional vectors (and tensors in general) are defined by their transformation properties in the **x**-space, we may define 4-vectors (and 4-tensors in general) by their transformation properties in the $(\mathbf{x}, t)$ space and find rules for their operation.

1. Any set of 4 quantities A^{μ} $(\mu = 0-3)$ or $\mathbf{A} = (A^0,\ A^1,\ A^2,\ A^3)$, which transform in the same way as x^{μ}, i.e.,

$$A'^{\mu} = \Lambda^{\mu}_{\nu} A^{\nu}, \tag{11.84'}$$

is called a 4-vector (or 4-tensor of the first rank).

The position vector $\mathbf{x}\ [\equiv (ct,\ x,\ y,\ z)]$ of a point in the 4-D space is obviously a 4-vector. As another example, the momentum vector of a particle in the 4-D space, defined as

$$\mathbf{p} \equiv (\frac{E}{c},\ p_x,\ p_y,\ p_z) = (\frac{E}{c},\ \mathbf{p}),$$

is a 4-vector because it transforms as [see Eq. (A.28), Appendix A.]

$$\begin{bmatrix} \frac{E'}{c} \\ p'_x \\ p'_y \\ p'_z \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{E}{c} \\ p_x \\ p_y \\ p_z \end{bmatrix} \quad \text{or} \quad p'^{\mu} = \Lambda^{\mu}_{\nu} p^{\nu}$$

2. If a quantity Φ is unchanged under the Lorentz transformation, it is called a Lorentz scalar (or 4-vector of the zeroth rank). The Lorentz scalar is also called a Lorentz invariant.

   The Lorentz scalar is in general a function of the components of a 4-vector. For example, we have just shown that

$$ds'^2 = dx'_\mu dx'^\mu = \frac{\partial x^\beta}{\partial x'^\mu} dx_\beta \frac{\partial x'^\mu}{\partial x^\lambda} dx^\lambda = \frac{\partial x^\beta}{\partial x'^\mu} \frac{\partial x'^\mu}{\partial x^\lambda} dx_\beta dx^\lambda$$

$$= \delta^\beta_\lambda dx_\beta dx^\lambda = ds^2$$

Hence, ds^2 is a Lorentz scalar.

3. Define the 4-D operator, as the counterpart of the operator ∇ in the **x**-space.

$$\begin{cases} \partial^\alpha \equiv \dfrac{\partial}{\partial x_\alpha} = (\frac{\partial}{\partial x^0}, \ -\nabla) \\ \partial_\alpha \equiv \dfrac{\partial}{\partial x^\alpha} = (\frac{\partial}{\partial x^0}, \ \nabla) \end{cases} \qquad (11.76)$$

$$\partial_\mu \Phi \equiv [\frac{\partial \Phi}{\partial x^0}, \ \frac{\partial \Phi}{\partial x}, \ \frac{\partial \Phi}{\partial y}, \ \frac{\partial \Phi}{\partial z}]$$

Then, the 4-gradient of a Lorentz scalar is a 4-vector.

$$Proof: \ \partial'_\mu \Phi = \frac{\partial \Phi}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial \Phi}{\partial x^\nu} = \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu \Phi = \Lambda^\nu_\mu \partial_\nu \Phi$$

Transforms as a 4-vector by (11.84).

4. The 4-divergence of a 4-vector, $\Box\cdot\mathbf{A}\equiv\frac{\partial A^\mu}{\partial x^\mu}$, is a Lorentz scalar.

Proof :

$$\Box\equiv\partial_\mu\equiv[\frac{\partial}{\partial x^0},\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}] \qquad \leftarrow \text{Griffiths (v)}$$

$$\Box\equiv\partial_\mu\partial^\mu=\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x^\alpha}=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2 \qquad \leftarrow \text{Jackson } (\times) \qquad (11.78)$$

$$\Box'\cdot\mathbf{A}'=\frac{\partial A'^\nu}{\partial x'^\nu}=\underbrace{\frac{\partial x^\mu}{\partial x'^\nu}\frac{\partial x'^\nu}{\partial x^\lambda}}_{=\delta^\mu_\lambda \text{ by (4)}}\frac{\partial A^\lambda}{\partial x^\mu}=\delta^\mu_\lambda\frac{\partial A^\lambda}{\partial x^\mu}=\frac{\partial A^\mu}{\partial x^\mu}=\Box\cdot\mathbf{A} \qquad (11.133')$$

$\Rightarrow$ $\Box\cdot\mathbf{A}$ is unchanged under the Lorentz transformation

Note: $A'^\alpha=\frac{\partial x'^\alpha}{\partial x^\beta}A^\beta$ and $B'_\alpha=\frac{\partial x^\beta}{\partial x'^\alpha}B_\beta$ (11.61&62)

5. The 4-Laplacian operator, $\Box^2 \equiv \partial^\alpha \partial_\alpha = \frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x^\alpha} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$,

is a Lorentz scalar operator, i.e., $\Box'^2 \Phi = \Box^2 \Phi$ [Φ : a Lorentz scalar].

Proof : We have shown in item 4 that the divergence of a 4-vector is a Lorentz scalar, i.e., $\Box' \cdot \mathbf{A}' = \Box \cdot \mathbf{A}$ or $\partial'_\nu A'^\nu = \partial_\mu A^\mu$.

Let Φ be a Lorentz scalar, then $A'^\nu = \partial'^\nu \Phi$ and $A^\mu = \partial^\mu \Phi$ are both 4-vectors (see item 3). Hence,

$$\Box' \cdot \mathbf{A}' = \partial'_\nu \partial'^\nu \Phi = \frac{\partial x^\lambda}{\partial x'^\nu}\partial_\lambda \frac{\partial x'^\nu}{\partial x^\beta}\partial^\beta \Phi = \delta^\lambda_\beta \partial_\lambda \partial^\beta \Phi = \partial_\beta A^\beta = \Box \cdot \mathbf{A}$$

$$\Rightarrow \quad \Box'^2 \Phi = \Box^2 \Phi.$$

$$\Box^2 \equiv \partial_\alpha \partial^\alpha = \partial^\alpha \partial_\alpha = \frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial x^\alpha} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \qquad (11.78')$$

6. The dot product of two 4-vectors, $\mathbf{A}\cdot\mathbf{B}\equiv A^{\mu}B_{\mu}$ (11.66), is a Lorentz scalar.

Proof : First recall $A'^{\alpha}=\Lambda^{\alpha}_{\beta}A^{\beta}=\dfrac{\partial x'^{\alpha}}{\partial x^{\beta}}A^{\beta}$, and $B'_{\alpha}=\dfrac{\partial x^{\beta}}{\partial x'^{\alpha}}B_{\beta}$

$$\mathbf{A}'\cdot\mathbf{B}'=A'^{\sigma}B'_{\sigma}=\overbrace{\frac{\partial x'^{\sigma}}{\partial x^{\nu}}A^{\nu}}^{A'^{\sigma}}\ \overbrace{\frac{\partial x^{\lambda}}{\partial x'^{\sigma}}B_{\lambda}}^{B'_{\sigma}}=\overbrace{\frac{\partial x^{\lambda}}{\partial x^{\nu}}}^{=\delta^{\lambda}_{\nu}\text{ by (4)}}A^{\nu}B_{\lambda}$$

$$=A^{\lambda}B_{\lambda}=\mathbf{A}\cdot\mathbf{B} \tag{11.66}$$

$$\mathbf{A}'\cdot\mathbf{B}'=A'^{\sigma}B'_{\sigma}=(g^{\sigma\alpha}A'_{\alpha})(g_{\beta\sigma}B'^{\beta})=\overbrace{(g^{\sigma\alpha}g_{\beta\sigma})}^{\delta^{\alpha}_{\beta}\ (11.71')}(A'_{\alpha}B'^{\beta})$$

$$=A'_{\sigma}B'^{\sigma}=\mathbf{A}\cdot\mathbf{B}=A^{\sigma}B_{\sigma}=A_{\sigma}B^{\sigma} \tag{11.66'}$$

11.1 Definitions and Operation Rules of … (*continued*)

7. A 4-tensor of the second rank ($\overleftrightarrow{\mathbf{F}}$) is a set of 16 quantities, $F'^{\alpha\beta}(\alpha,\beta=0-3)$, which transform according to

$$F'^{\alpha\beta}=\frac{\partial x'^{\alpha}}{\partial x^{\gamma}}\frac{\partial x'^{\beta}}{\partial x^{\delta}}F^{\gamma\delta}=\Lambda^{\alpha}_{\gamma}\Lambda^{\beta}_{\delta}F^{\gamma\delta} \qquad (11.63)$$

Contraction with $g^{\alpha\beta}$ and $g_{\alpha\beta}$ is the procedure for changing an index on any tensor from being contravariant to covariant, and vice versa.

$$\begin{aligned} F^{\cdot\cdot\alpha\cdot\cdot}_{\cdot\cdot} &= g^{\alpha\beta}F^{\cdot\cdot\cdot\cdot}_{\cdot\cdot\beta} \\ G^{\cdot\cdot\cdot\cdot}_{\cdot\cdot\cdot\alpha\cdot} &= g_{\alpha\beta}G^{\cdot\cdot\cdot\beta}_{\cdot\cdot\cdot\cdot} \end{aligned} \qquad (11.74)$$

The covariant 4-vector x_{α} can be obtain from the contravariant x^{α} by contraction with $g_{\alpha\beta}$ and its inverse.

$$x_{\alpha}=g_{\alpha\beta}x^{\beta} \qquad (11.72)$$

$$x^{\alpha}=g^{\alpha\beta}x_{\beta} \qquad (11.73)$$

11.1 Definitions and Operation Rules of … (*continued*)

8. The dot product of a 4-tensor of the second rank and a 4-vector,

$(\overleftrightarrow{\mathbf{T}}\cdot\mathbf{A})^{\mu} \equiv T^{\mu\nu}A_{\nu}$, is a 4-vector.

$$\mathit{Proof}:\ (\overleftrightarrow{\mathbf{T}}'\cdot\mathbf{A}')^{\mu} = T'^{\mu\nu}A'_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\gamma}}\frac{\partial x'^{\nu}}{\partial x^{\delta}}T^{\gamma\delta}\frac{\partial x^{\alpha}}{\partial x'^{\nu}}A_{\alpha}$$

$$= \frac{\partial x'^{\mu}}{\partial x^{\gamma}}\underbrace{\left(\frac{\partial x'^{\nu}}{\partial x^{\delta}}\frac{\partial x^{\alpha}}{\partial x'^{\nu}}\right)}_{\delta^{\alpha}_{\delta}}T^{\gamma\delta}A_{\alpha} = \frac{\partial x'^{\mu}}{\partial x^{\gamma}}\underbrace{T^{\gamma\delta}A_{\delta}}_{(\overleftrightarrow{\mathbf{T}}\cdot\mathbf{A})^{\gamma}}$$

It transforms as a 4-vector.

$$T^{\mu\nu}A_{\nu} = (g^{\nu\sigma}T^{\mu}_{\sigma})(g_{\alpha\nu}A^{\alpha}) = \delta^{\sigma}_{\alpha}T^{\mu}_{\sigma}A^{\alpha} = T^{\mu}_{\sigma}A^{\sigma}$$

Matrix representation: $\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}_{4\times 4}\begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}_{4\times 1} = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}_{4\times 1}$

11.1 Definitions and Operation Rules of … (*continued*)

9. The 4-divergence of a 4-tensor of the second rank, $(\Box'\cdot\overleftrightarrow{\mathbf{T}}')^{\mu} \equiv \dfrac{\partial T'^{\mu\nu}}{\partial x'^{\nu}}$ or $(\Box\cdot\overleftrightarrow{\mathbf{T}})_{\mu} \equiv \dfrac{\partial T_{\mu\nu}}{\partial x_{\nu}}$ is a 4-vector.

Proof :

$$(\Box'\cdot\overleftrightarrow{\mathbf{T}}')^{\mu} = \frac{\partial T'^{\mu\nu}}{\partial x'^{\nu}} = \left(\frac{\partial x^{\alpha}}{\partial x'^{\nu}}\frac{\partial}{\partial x^{\alpha}}\right)\left(\frac{\partial x'^{\mu}}{\partial x^{\gamma}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}}T^{\gamma\sigma}\right)$$

Transform as a 4-vector.

$$= \underbrace{\left(\frac{\partial x^{\alpha}}{\partial x'^{\nu}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}}\right)}_{\delta^{\alpha}_{\sigma}}\left(\frac{\partial x'^{\mu}}{\partial x^{\gamma}}\frac{\partial}{\partial x^{\alpha}}T^{\gamma\sigma}\right) = \left(\frac{\partial x'^{\mu}}{\partial x^{\gamma}}\frac{\partial}{\partial x^{\alpha}}T^{\gamma\alpha}\right) = \frac{\partial x'^{\mu}}{\partial x^{\gamma}}(\Box\cdot\overleftrightarrow{\mathbf{T}})^{\gamma} \qquad (11.142')$$

$$(\Box'\cdot\overleftrightarrow{\mathbf{T}}')_{\mu} = \frac{\partial T'_{\mu\nu}}{\partial x'_{\nu}} = \left(\frac{\partial x'^{\nu}}{\partial x^{\alpha}}\frac{\partial}{\partial x_{\alpha}}\right)\left(\frac{\partial x^{\gamma}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}T_{\gamma\sigma}\right)$$

$$= \underbrace{\left(\frac{\partial x'^{\nu}}{\partial x^{\alpha}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\right)}_{\delta^{\sigma}_{\alpha}}\left(\frac{\partial x^{\gamma}}{\partial x'^{\mu}}\frac{\partial}{\partial x_{\alpha}}T_{\gamma\sigma}\right) = \left(\frac{\partial x^{\gamma}}{\partial x'^{\mu}}\frac{\partial}{\partial x_{\alpha}}T_{\gamma\alpha}\right) = \frac{\partial x^{\gamma}}{\partial x'^{\mu}}(\Box\cdot\overleftrightarrow{\mathbf{T}})_{\gamma} \qquad (11.143')$$

11.1 Definitions and Operation Rules of … *(continued)*

$$(\Box\cdot\overleftrightarrow{\mathbf{T}})_{\mu} \equiv \frac{\partial T_{\mu\nu}}{\partial x_{\nu}} = \partial^{\nu} T_{\mu\nu} \quad \Rightarrow \quad \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}_{4\times1} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}_{4\times4} \neq (\cdot \quad \cdot \quad \cdot \quad \cdot)_{1\times4}$$

Why?

The format in matrix representation is inconsistent. How to solve it?

Sol: Rewrite the format

$$(\Box\cdot\overleftrightarrow{\mathbf{T}})_{\mu} \equiv \frac{\partial T_{\mu\nu}}{\partial x_{\nu}} = \partial^{\nu} T_{\mu\nu} \;=\; \partial_{\nu} T_{\mu}^{\nu}$$

$$\text{so} \quad (\cdot \quad \cdot \quad \cdot \quad \cdot)_{1\times4} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}_{4\times4} = (\cdot \quad \cdot \quad \cdot \quad \cdot)_{1\times4}$$

Transform as a 4-vector.

Extra point!

Derive the correct form for $(\Box\cdot\overleftrightarrow{\mathbf{T}})^{\mu} \equiv \partial_{\nu} T^{\mu\nu}$ with the same approach.

10. A 4-tensor of the third rank is a set of 64 quantities, $G^{\lambda\mu\nu}(\lambda,\mu,\nu=0\text{-}3)$, which transform according to

$$G'^{\lambda\mu\nu}=\frac{\partial x'^{\lambda}}{\partial x^{i}}\frac{\partial x'^{\mu}}{\partial x^{j}}\frac{\partial x'^{\nu}}{\partial x^{k}}G^{ijk}$$

Problem 1: If $F^{\mu\nu}$ is a 4-tensor of the second rank, show that

$$\frac{\partial F^{\mu\nu}}{\partial x_{\lambda}}(\lambda,\mu,\nu=0\text{-}3)\text{ is a 4-tensor of the third rank.}$$

$$Solution:\quad F'^{\mu\nu}=\frac{\partial x'^{\mu}}{\partial x^{j}}\frac{\partial x'^{\nu}}{\partial x^{k}}F^{jk}$$

$$\Rightarrow\frac{\partial F'^{\mu\nu}}{\partial x'_{\lambda}}=\partial'^{\lambda}\Big(\frac{\partial x'^{\mu}}{\partial x^{j}}\frac{\partial x'^{\nu}}{\partial x^{k}}F^{jk}\Big)=\frac{\partial x'^{\lambda}}{\partial x^{i}}\partial^{i}\Big(\frac{\partial x'^{\mu}}{\partial x^{j}}\frac{\partial x'^{\nu}}{\partial x^{k}}F^{jk}\Big)$$

$$=\frac{\partial x'^{\lambda}}{\partial x^{i}}\frac{\partial x'^{\mu}}{\partial x^{j}}\frac{\partial x'^{\nu}}{\partial x^{k}}\Big(\frac{\partial F_{jk}}{\partial x_{i}}\Big)$$

Transform as a 4−tensor of the third rank.

11.1 Definitions and Operation Rules of … (*continued*)

Problem 2 : Show that the set of equations,

$$\frac{\partial F_{\mu\nu}}{\partial x^{\lambda}}+\frac{\partial F_{\lambda\mu}}{\partial x^{\nu}}+\frac{\partial F_{\nu\lambda}}{\partial x^{\mu}}=0\ (\lambda,\mu,\nu=0\text{-}3) \qquad (11.143)$$

is invariant in form under the Lorentz transformation.

Solution :

$$\frac{\partial F'_{\mu\nu}}{\partial x'^{\lambda}}=\partial'_{\lambda}F'_{\mu\nu}=\left(\frac{\partial x^{i}}{\partial x'^{\lambda}}\partial_{i}\right)\left(\frac{\partial x^{j}}{\partial x'^{\mu}}\frac{\partial x^{k}}{\partial x'^{\nu}}F_{jk}\right) \qquad (11.64')$$

$$=\left(\frac{\partial x^{i}}{\partial x'^{\lambda}}\frac{\partial x^{j}}{\partial x'^{\mu}}\frac{\partial x^{k}}{\partial x'^{\nu}}\right)\partial_{i}F_{jk}$$

$$\frac{\partial F'_{\mu\nu}}{\partial x'^{\lambda}}+\frac{\partial F'_{\lambda\mu}}{\partial x'^{\nu}}+\frac{\partial F'_{\nu\lambda}}{\partial x'^{\mu}}=\left(\frac{\partial x^{i}}{\partial x'^{\lambda}}\frac{\partial x^{j}}{\partial x'^{\mu}}\frac{\partial x^{k}}{\partial x'^{\nu}}\right)\overbrace{(\partial_{i}F_{jk}+\partial_{j}F_{ki}+\partial_{k}F_{ij})}^{\frac{\partial F_{\mu\nu}}{\partial x^{\lambda}}+\frac{\partial F_{\lambda\mu}}{\partial x^{\nu}}+\frac{\partial F_{\nu\lambda}}{\partial x^{\mu}}=0}=0$$

It is invariant in form under the Lorentz transformation.

11. If a physical law can be expressed as a relation between 4-tensors of the same rank, then its form is invariant in all Lorentz frames.

Example 1: If the physical law in frame K is of the form $\mathbf{A}=\mathbf{B}$,

$$\text{then, } A'_\nu = \frac{\partial x'^\mu}{\partial x^\nu}\underbrace{A_\mu}_{B_\mu} = \frac{\partial x'^\mu}{\partial x^\nu}B_\mu = B'_\nu, \text{ i.e., } \mathbf{A}=\mathbf{B}\Rightarrow \mathbf{A}'=\mathbf{B}'.$$

Example 2: If the physical law in frame K is of the form $\overleftrightarrow{\mathbf{T}}=\overleftrightarrow{\mathbf{F}}$,

$$\text{then, } T'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^i}\frac{\partial x'^\nu}{\partial x^j}\underbrace{T^{ij}}_{F^{ij}} = \frac{\partial x'^\mu}{\partial x^i}\frac{\partial x'^\nu}{\partial x^j}F^{ij} = F'^{\mu\nu}, \text{ i.e.,}$$

$$\overleftrightarrow{\mathbf{T}}=\overleftrightarrow{\mathbf{F}}\Rightarrow \overleftrightarrow{\mathbf{T}}'=\overleftrightarrow{\mathbf{F}}' \quad \text{[i.e., invariant in form]}$$

In the following section, we examine relativistic mechanics in 4-vector formalism. In Sec. 3, we will demonstrate that laws of electromagnetism are invariant under the Lorentz transformation by expressing them as relations between tensors of the same rank. From the Lorentz transformation of these tensors, we also obtain the transformation equations for various electromagnetic quantities.

Section 2: Relativistic Mechanics

We begin with a note on the terms "conservation", "invariance", and "covariance".

The conservation of a quantity means that it remains unchanged in time in a given Lorentz frame. For example, the relativistic momentum and energy of an isolated system of particles are both conserved after a collision. This is a fundamental law to be discussed in this Section.

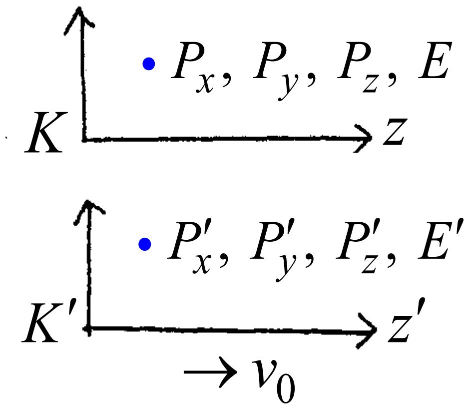
The invariance of a quantity means that it is invariant in value under a Lorentz transformation. Such a quantity is called a Lorentz invariant or Lorentz scalar. For example, the dot product of two 4-vectors is a Lorentz invariant. However, it may or may not be a conserved quantity. An example will be provided in this section.

The term covariance refers to physical laws. A physical law is "covariant" if it is "invariant in form under the Lorentz transformation." As will be shown, the new laws of relativistic mechanics and existing laws of electromagnetism are all covariant.

The 4-Momentum (p) of a Single Particle:

As shown in (A.28), if we define the momentum of a particle as $\mathbf{p} \equiv \gamma m \mathbf{v}$ and energy as $E \equiv \gamma m c^2$ (m is called the <u>rest mass</u>*), then the <u>4-momentum</u>, $\mathsf{p} \equiv (\frac{E}{c}, p_x, p_y, p_z)$, is a 4-vector, which transforms as

$$\begin{cases} p'_x = p_x \\ p'_y = p_y \\ p'_z = \gamma_0 (p_z - \frac{v_0}{c^2} E) \\ E' = \gamma_0 (E - v_0 p_z) \end{cases}$$



*Throughout this chapter, m and M denote the rest mass.

Discussion: In Appendix A, we first define $\mathbf{p} = \gamma m \mathbf{v}$ and $E = \gamma m c^2$, then show that the law of conservation of momentum and energy is covariant. Conversely, from the requirement of the covariance of this conservation law, we can deduce the definitions of $\mathbf{p} = \gamma m \mathbf{v}$ and $E = \gamma m c^2$ (see Jackson Sec. 11.5).

The dot product of two 4-vectors is a Lorentz scalar, hence

$$\mathbf{p}\cdot\mathbf{p}=\mathbf{p}'\cdot\mathbf{p}' \;\Rightarrow\; p^2-\frac{E^2}{c^2}=p'^2-\frac{E'^2}{c^2} \qquad (11.54)$$

i.e., $E^2-p^2c^2$ is a Lorentz scalar (invariant).

If frame K' is the rest frame of the particle (i.e., $p'=0,\ E'=mc^2$) then $\mathbf{p}'=(mc,\ 0,\ 0,\ 0)$ and $\mathbf{p}\cdot\mathbf{p}=\mathbf{p}'\cdot\mathbf{p}'$ gives $\frac{E^2}{c^2}-p^2=m^2c^2$, or

$$E^2-p^2c^2=m^2c^4 \qquad (11.54)$$

Since $E^2-p^2c^2$ is a Lorentz invariant, (11.54) shows that the rest mass m is a Lorentz invariant. This has in fact been assumed in Sec. 2 of Appendix A, where we derive the Lorentz transformation equations for $\mathbf{p}$ $(=\gamma m\mathbf{v})$ and E $(=\gamma mc^2)$. (11.54) is a useful formula for it relates the particle's total energy (E) to its momentum (p). (Momentum in particle physics is often expressed in unit of GeV/c.)

For a relativistic particle, we can still speak of its (macroscopic) kinetic energy K, defined as: $K=E-mc^2=(\gamma-1)mc^2$.

The 4 - Momentum ($\mathbf{P}$) of a System of Particles

Consider a system of particles, each with the 4-momentum

$$\mathbf{p}_j = (E_j / c, p_{xj}, p_{yj}, p_{zj}) = (E_j / c, \mathbf{p}_j), \; j = 1,\; 2,\; 3, \cdots$$

Since the Lorentz transformation is a linear transformation, the sum of any number of 4-vectors also obeys the Lorentz tansformation. Thus, $\mathbf{P} = \sum_j \mathbf{p}_j$ is a 4-vector and its components transform as

$$\begin{cases} \sum_j p'_{xj} = \sum_j p_{xj} \\ \sum_j p'_{yj} = \sum_j p_{yj} \\ \sum_j p'_{zj} = \gamma_0 (\sum_j p_{zj} - \frac{v_0}{c^2} \sum_j E_j) \\ \sum_j E'_j = \gamma_0 (\sum_j E_j - v_0 \sum_j p_{zj}) \end{cases}$$

and $\mathbf{P} \cdot \mathbf{P} = (\sum_j \mathbf{p}_j) \cdot (\sum_j \mathbf{p}_j) = (\sum_j \mathbf{p}_j) \cdot (\sum_j \mathbf{p}_j) - (\sum_j E_j / c)^2 \qquad (11.54')$

is a Lorentz invariant.

Law of Conservation of Momentum and Energy :

In relativity, the conservation of momentum and energy comes in one law rather than separate laws for the momentum and energy as in nonrelativistic mechanics. The law states that, for an *isolated* system of particles,

$$\mathbf{P}(\text{before collision}) = \mathbf{P}(\text{after collision}),$$

which implies that $\sum_j p_{xj}$, $\sum_j p_{yj}$, $\sum_j p_{zj}$, and $\sum_j E_j$ are each conserved, i.e.,

$$\sum_j \mathbf{p}_j(\text{before collision}) = \sum_j \mathbf{p}_j(\text{after collision})$$

$$\sum_j E_j(\text{before collision}) = \sum_j E_j(\text{after collision})$$

Since the conservation law is expressed as a 4-vector relation, it has the same form in all Lorentz frames. Thus, in frame K', we have $\mathbf{P}'(\text{before collision}) = \mathbf{P}'(\text{after collision})$.

If $\mathbf{P}$ is conserved, the dot product $\mathbf{P}\cdot\mathbf{P}$ must also be conserved. Thus,

$$\underbrace{(\sum_j \mathbf{p}_j)\cdot(\sum_j \mathbf{p}_j)-(\sum_j \frac{E_j}{c})^2}_{\text{before collision}}=\underbrace{(\sum_j \mathbf{p}_j)\cdot(\sum_j \mathbf{p}_j)-(\sum_j \frac{E_j}{c})^2}_{\text{after collision}} \qquad (11.54')$$

Discussion:

(i) $\mathbf{P}\cdot\mathbf{P}$ for an *isolated* system is both a Lorentz invariant and a conserved quantity. If the system is not isolated, it is still a Lorentz invariant, but no longer a conserved quantity.

(ii) $\mathbf{P}(\text{before collision})=\mathbf{P}(\text{after collision})$ is a fundamental law (rather than a derived relation), in which the nonrelativistic law of conservation of momentum has been extended to include the energy, $E=\gamma mc^2$. A very important aspect of this law is that it applies to all processes in an isolated system, such as elastic and inelastic collisions, nuclear reactions, and particle decays. As a result, the total rest mass of the system may not be conserved, as is illustrated in the following two problems.

11.2 Relativistic Mechanics (*continued*)

Problem 1: Two identical particles of rest mass m and equal and opposite velocities $\pm\mathbf{v}$ collide head-on inelastically to form a single particle. Find the mass and velocity of the new particle.

Solution:

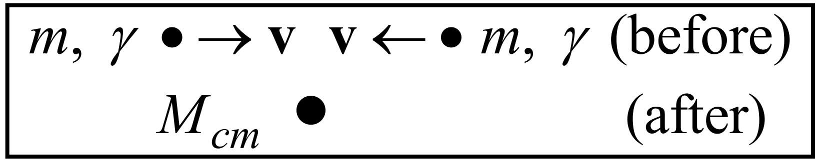
$$\boxed{\begin{array}{ll} m,\ \gamma \bullet\!\rightarrow \mathbf{v} \quad \mathbf{v} \leftarrow\!\bullet\ m,\ \gamma & \text{(before)} \\ \qquad M_{cm} \quad \bullet & \text{(after)} \end{array}}$$

The total momentum before the collision is $\gamma m\mathbf{v} - \gamma m\mathbf{v} = 0$.

So the collision occurs in the center-of-momentum (CM) frame, i.e., the frame in which the sum of the momentum of all particles vanishes. For later comparison with the result in problem 2, we denote the mass of the new particle by M_{cm} to indicate that it is created in the CM frame.

$$\begin{cases} \text{Conservation of momentum} \Rightarrow \text{The new particle is stationary.} \\ \text{Conservation of energy} \qquad \Rightarrow \gamma m + \gamma m = M_{cm} \Rightarrow \ M_{cm} = 2\gamma m \end{cases}$$

11.2 Relativistic Mechanics (*continued*)



Discussion: In this problem, we find $M_{cm} = 2\gamma m > 2m$, i.e. rest mass has been created from the kinetic energy [$(\gamma - 1)mc^2$] of the colliding particles. There is no need to know what's inside the new particle. We only need to know its rest mass and hence the energy associated with it. A hot object has a rest mass greater than when it's cold. The difference in rest mass due to an increase in temperature can in principle be measured by its acceleration under a known force, and we know that at least some of the added mass is in the form of thermal energy. In many other cases, it's not possible to know what's inside.

Nuclear fusion and fission reactions are examples of non-conservation of rest mass. The total rest mass is reduced after the reaction and the mass deficit appears as kinetic energies and radiation. In fact, all reactions (chemical or nuclear) in which energy is absorbed (e.g., photosynthesis) or released (e.g., digestion of food) involve a corresponding change of the reactants' total rest mass.

11.2 Relativistic Mechanics (*continued*)

Problem 2: A particle of rest mass m and velocity $\mathbf{v}$ collides with a stationary particle of the same rest mass and is absorbed by it. Find the rest mass and velocity of the new particle.

Solution: The collision occurs in the stationary-target (ST) frame. So, we denote the new particle mass by M_{st}, velocity by $\mathbf{V}_{st}$, and Lorentz factor by γ_{st} $[=(1-V_{st}^2/c^2)^{-1/2}]$. ($m, \gamma, \mathbf{v}$ are also ST frame quantities.)

$$\begin{cases} \text{Conservation of momentum} \Rightarrow \gamma m \mathbf{v} = \gamma_{st} M_{st} \mathbf{V}_{st} & (1) \\ \text{Conservation of energy} \Rightarrow (\gamma+1)m = \gamma_{st} M_{st} & (2) \end{cases}$$

$$\frac{(1)}{(2)} \Rightarrow \mathbf{V}_{st} = \frac{\gamma}{\gamma+1}\mathbf{v}$$

$$(2) \Rightarrow M_{st} = \frac{\gamma+1}{\gamma_{st}} m$$

$$\boxed{\begin{array}{ll} m,\ \gamma \bullet \longrightarrow \mathbf{v} \quad \bullet\ m & \text{(before)} \\ M_{st},\ \gamma_{st} \bullet \longrightarrow \mathbf{V}_{st} & \text{(after)} \end{array}}$$

$$\Rightarrow M_{st}^2 = m^2 \frac{(\gamma+1)^2}{\gamma_{st}^2} = m^2(\gamma+1)^2\left(1-\frac{V_{st}^2}{c^2}\right)^2 = m^2(\gamma+1)^2\left[1-\frac{\gamma^2 v^2}{c^2(\gamma+1)^2}\right]$$

$$= m^2\left(\gamma^2+2\gamma+1-\gamma^2\frac{v^2}{c^2}\right) = m^2\left[\gamma^2\left(1-\frac{v^2}{c^2}\right)+2\gamma+1\right] = 2m^2(\gamma+1)$$

$$\Rightarrow M_{st} = \sqrt{2(\gamma+1)}m$$

11.2 Relativistic Mechanics (*continued*)

Discussion :

In problem 1 (CM frame), the new particle's mass is $M_{cm} = 2\gamma m.$ (33)

$$\boxed{\begin{array}{l} m,\ \gamma \bullet \rightarrow \mathbf{v} \quad \mathbf{v} \leftarrow \bullet\ m,\ \gamma \ \text{(before)} \\ \qquad M_{cm}\ \bullet \qquad\qquad\qquad \text{(after)} \end{array}}$$

In problem 2 (ST frame), the new particle's mass is $M_{st} = \sqrt{2(1+\gamma)}m.$ (34)

$$\boxed{\begin{array}{l} m,\ \gamma \bullet \longrightarrow \mathbf{v} \quad \bullet\ m \quad \text{(before)} \\ M_{st},\ \gamma_{st}\ \bullet \longrightarrow \mathbf{V}_{st} \quad \text{(after)} \end{array}}$$

Note that γ is the Lorentz factor of the particle(s) before collision.

In particle physics experiments, $M_{cm}c^2$ or $M_{st}c^2$ is the energy available for the creation of new particles (why not $\gamma_{st}M_{st}c^2$?).

The rest energy of the electron or positron is $mc^2 = 0.511$ MeV. If 2 TeV of energy is needed for particle creation (i.e., $M_{cm}c^2 = 2$ TeV or $M_{st}c^2 = 2$ TeV), then the required γ of the colliding particle(s) is

$$\begin{cases} \text{by (33), } M_{cm}c^2 = 2\gamma mc^2 = 2 \text{ TeV} \Rightarrow \gamma \approx 1.957\times 10^6 & \text{[CM frame]} \\ \text{by (34), } M_{st}c^2 = \sqrt{2(1+\gamma)}mc^2 = 2 \text{ TeV} \Rightarrow \gamma \approx 7.66\times 10^{12} & \text{[ST frame]} \end{cases}.$$

The energy associated with γ is to be obtained in an accelerator.

Thus,

$$\frac{\text{kinetic energy needed in CM frame}}{\text{kinetic energy needed in ST frame}} = \frac{2\times(1.957\times10^{6}-1)}{7.66\times10^{12}-1} \approx 5\times10^{-7}$$

This shows that far less kinetic energy is needed in the CM frame than in the ST frame. In fact, all the kinetic energy of the two colliding particles $[2\times(1.957\times10^{6}-1)\times0.511\text{ MeV} = 2\text{ TeV}]$ is put in use in the CM frame, while in the ST frame, 99.99995% of the kinetic energy of the incident particle is wasted! This is why the International Linear Collider (ILC) project plans to accelerate both electrons and positirons to energies up to 1 TeV so that the the collision occurs in the CM frame.

Question: Why use a long linear accelerator instead of a more compact circular accelerator?

Section 3: Covariance of Electrodynamics

In the special theory of relativity, Newton's law has been radically modified. The electromagnetic laws do not need any modification because they are already covariant. However, the *covariance* of these laws (such as Maxwell equations) is not immediately clear from the equations by which they are usually represented.

Our purpose in this section is to prove that the EM laws are indeed covariant by casting them into relations between 4-tensors of the same rank. We will do this by first defining 4-tensors in terms of known EM quantities and forming equations with 4-tensors of the same rank, then show that one or more existing EM laws are implicit in each equation. This will prove that the laws are convariant and justify the defined quantities to be legitimate 4-tensors.

Furthermore, Lorentz transformations of these tensors will yield the tranformation equations for various EM quantities.

Note: Jackson switches to the Gausian unit system starting from Ch. 11. From here on, we also adopt the Gaussian unit system.

11.3 Covariance of Electrodynamics (*continued*)

SI unit $\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \quad (6.3)$

1. Define a 4-current as

$$\mathbf{J} \equiv J^{\nu} = (c\rho, J_x,\ J_y,\ J_z) = (c\rho, \mathbf{J}) \qquad (11.128)$$

and use it to form a relation

The invariance of electric charge under Lorentz transformation

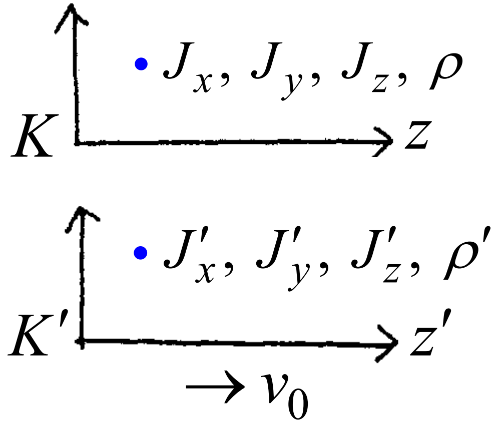
$$\Box \cdot \mathbf{J} = \partial_{\nu} J^{\nu} = \partial^{\nu} J_{\nu} = 0 \qquad (11.129)$$

Then, (36) gives the law of conservation of charge

$$\frac{\partial}{\partial x} J_x + \frac{\partial}{\partial y} J_y + \frac{\partial}{\partial z} J_z + \frac{\partial (c\rho)}{\partial (ct)} = 0 \ \Rightarrow \ \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \qquad (11.127)$$

Thus, the definition of **J** as a 4-vector leads to the covariant representation of the EM law in (5.2). This in turn justifies the definition of **J** as a 4-vector. The Lorentz transformation of **J** then gives

$$\begin{cases} J'_x = J_x \\ J'_y = J_y \\ J'_z = \gamma_0 (J_z - v_0 \rho) \\ \rho' = \gamma_0 \left(\rho - \frac{v_0}{c^2} J_z\right) \end{cases} \qquad (37)$$



11.3 Covariance of Electrodynamics (*continued*)

$\boxed{(\frac{V}{c}, A_x, A_y, A_z) \leftarrow \text{Griffiths}}$

2. Define a 4-potential as $A^\alpha \equiv (\Phi, A_x, A_y, A_z)$ (11.132)

and write the covariant relations:

$$\boxed{\text{SI unit (6.15\&16)} \quad \begin{cases} \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi - \nabla^2\Phi = \frac{\rho}{\varepsilon_0} \\ \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} - \nabla^2\mathbf{A} = \mu_0\mathbf{J} \end{cases}}$$

d'Alembertian

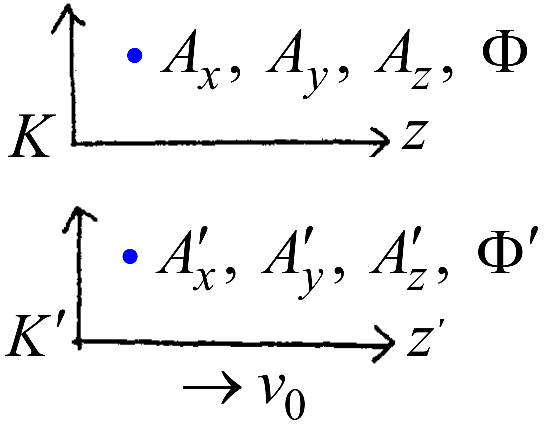
$$\Box^2 \mathbf{A} = \partial^\nu \partial_\nu A^\alpha = \frac{4\pi}{c} J^\alpha \Rightarrow \begin{cases} \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} - \nabla^2\mathbf{A} = \frac{4\pi}{c}\mathbf{J} \\ \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi - \nabla^2\Phi = 4\pi\rho \end{cases} \qquad (11.130)$$

$$\Box \cdot \mathbf{A} = \partial_\nu A^\nu = 0 \qquad \Rightarrow \quad \nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial}{\partial t}\Phi = 0 \qquad (11.131)$$

$$\boxed{\nabla\cdot\mathbf{A} + \mu_0\varepsilon_0\frac{\partial V}{\partial t} = 0 \quad \leftarrow \text{Griffiths}}$$

This again shows the consistency of **A** being a 4-vector and (11.130) and (11.131) being covariant laws. The Lorentz transformation

of **A** then gives
$$\begin{cases} A'_x = A_x \\ A'_y = A_y \\ A'_z = \gamma_0\left(A_z - \frac{v_0}{c}\Phi\right) \\ \Phi' = \gamma_0\left(\Phi - \frac{v_0}{c}A_z\right) \end{cases} \qquad (41)$$



11.3 Covariance of Electrodynamics (*continued*)

Note: The source-free wave equation can be directly put into the covariant form: $\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = 0 \;\Rightarrow\; \Box^2\psi = 0.$

3. Define a 4-wavenumber as

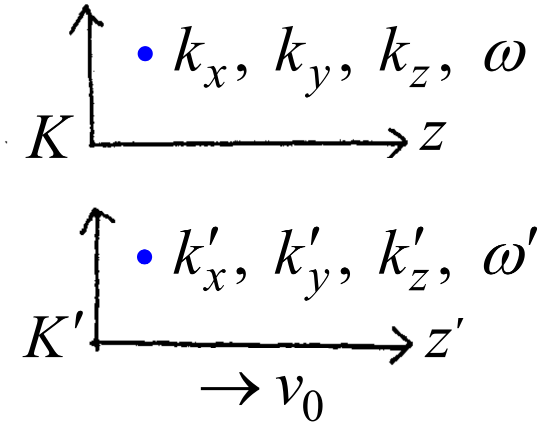
$$\mathbf{k} \equiv \left(\tfrac{\omega}{c},\, k_x,\, k_y,\, k_z\right) = \left(\tfrac{\omega}{c},\, \mathbf{k}\right)$$

Then, $\mathbf{k}' \cdot \mathbf{x}' = \mathbf{k} \cdot \mathbf{x} \Rightarrow \;\omega' t' - \mathbf{k}' \cdot \mathbf{x}' = \omega t - \mathbf{k} \cdot \mathbf{x}$ (11.28)

⇒ Invariance of the phase

By the same argument, we find that **k** defined in (43) is a legitimate 4-vector. Thus, its Lorentz transformation gives

$$\underbrace{\begin{cases} k_x' = k_x \\ k_y' = k_y \\ k_z' = \gamma_0\left(k_z - \frac{v_0}{c^2}\omega\right) \\ \omega' = \gamma_0(\omega - v_0 k_z) \end{cases}}_{\text{Relativistic Doppler shift}} \qquad (11.29)$$



4. Define a field strength tensor of the second rank $F^{\alpha\beta}$ and $F_{\alpha\beta}$ the elements of a second-rank, antisymmetric field-strength tensor.

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = -F^{\beta\alpha} \qquad (11.136)$$

$$F^{\alpha\beta} \equiv \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \qquad (11.137)$$

$$F_{\alpha\beta} = g_{\alpha\gamma} F^{\gamma\delta} g_{\delta\beta} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix} \qquad (11.138)$$

The elements of $F_{\alpha\beta}$ are obtained from $F^{\alpha\beta}$ by putting $\mathbf{E} \to -\mathbf{E}$.

* The inhomogeneous equations are:

$$\Box\cdot \overset{\leftrightarrow}{\mathbf{F}} = \partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta \;\Rightarrow\; \begin{cases} \nabla\cdot\mathbf{E} = 4\pi\rho \\ \nabla\times\mathbf{B} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J} \end{cases} \qquad (11.141)$$

These two equations are covariant.

* The homogeneous equations are: See p. 25

$$\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} = 0 \;(\lambda,\mu,\nu = 0\text{-}3) \qquad (11.143)$$

set $(\lambda,\mu,\nu) = (1,2,3) \Rightarrow \nabla\cdot\mathbf{B} = 0$

set $(\lambda,\mu,\nu) = (0,1,2),\ (0,1,3),$ and $(0,2,3) \Rightarrow \nabla\times\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} = 0.$

SI unit $\begin{cases} \nabla\cdot\mathbf{D} = \rho \\ \nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t} = \mathbf{J} \end{cases}$ and $\begin{cases} \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0 \end{cases}$

11.3 Covariance of Electrodynamics (*continued*)

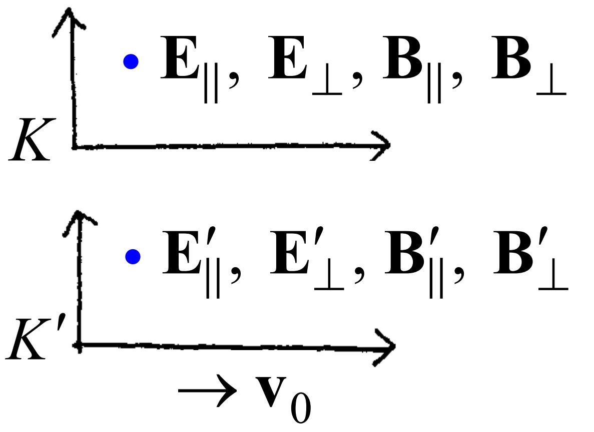
The covariant equations, $\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta$ and $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$, give the set of Maxwell equations in free space. This shows that Maxwell equations are covariant as well as justify the definition of $\overleftrightarrow{\mathbf{F}}$ as a tensor of the second rank. Thus, $F'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial x'^\beta}{\partial x^\delta} F^{\gamma\delta}$ (11.146)

gives the transformation equations for **E** and **B**.

In the matrix notation, this can be written $F' = AF\tilde{A}$, (11.147)

where A is the Lorentz transformation matrix.

$$\begin{cases} \mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \\ \mathbf{E}'_{\perp} = \gamma_0 \left(\mathbf{E}_{\perp} + \frac{\mathbf{v}_0}{c} \times \mathbf{B}_{\perp} \right) \\ \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel} \\ \mathbf{B}'_{\perp} = \gamma_0 \left(\mathbf{B}_{\perp} - \frac{\mathbf{v}_0}{c} \times \mathbf{E}_{\perp} \right) \end{cases} \tag{11.149'}$$



In (46), $\mathbf{v}_0$ is the velocity of frame K' relative to frame K, and "$\parallel$" and "$\perp$" refer to the direction of $\mathbf{v}_0$ (See Appendix C).

11.3 Covariance of Electrodynamics (*continued*)

5. The covariant equation*, $\frac{d}{d\tau}P^{\alpha} = \frac{e}{mc}F^{\alpha}_{\beta}P^{\beta} = \frac{e}{mc}F^{\alpha\beta}P_{\beta}$ ($d\tau$ is a Lorentz scalar, see Appendix A Time Dilation), gives: $\frac{d}{dt}P^{\alpha} = \frac{d\tau}{dt}\frac{e}{mc}F^{\alpha\beta}P_{\beta}$,

where $P^{\alpha} \equiv (\frac{E}{c}, \mathbf{p})$, $P_{\beta} \equiv (\frac{E}{c}, -\mathbf{p})$, $\mathbf{p} \equiv \gamma m \mathbf{v}$, and $\frac{d\tau}{dt} = \frac{1}{\gamma}$.

$$F^{\alpha\beta} \equiv \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

$$\frac{d}{dt}p_x = \frac{e}{\gamma mc}\left(E_x\frac{E}{c} + \gamma m(v_y B_z - v_z B_y)\right) = e\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)_x$$

$$\begin{cases} \frac{d}{dt}\mathbf{p} = e\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right) & \begin{bmatrix} \alpha = 1,2,3 \text{ relativistic} \\ \text{equation of motion} \end{bmatrix} \qquad (11.124) \\ mc\frac{d}{dt}\gamma = e\mathbf{v}\cdot\mathbf{E} & \begin{bmatrix} \alpha = 0 \text{ work only} \\ \text{done by } E\text{-field} \end{bmatrix} \qquad (11.126) \end{cases}$$

*In order for this equation to be covariant, the charge e must be a Lorentz invariant. This has been experimentally established (see Jackson, p.554).

6. In a similar manner, we can demonstrate the covariance of the conservation laws for field/mechanical energy and field/mechanical momentum, as given by Jackson (6.111) and (6.122):

$$\begin{cases} \frac{d}{dt}\left(E_{\text{mech}} + E_{\text{field}}\right) = -\oint_S \mathbf{n}\cdot\mathbf{S}da & (6.111) \\ \frac{d}{dt}\left(\mathbf{p}_{\text{mech}} + \mathbf{p}_{\text{field}}\right) = \oint_S \sum_\beta T_{\alpha\beta} n_\beta da & (6.122) \end{cases}$$

Proof: If a physical law can be expressed as a relation between 4-tensors of the same rank, then its form is invariant in all Lorentz frames.

First, define the symmetric stress tensor

$$\Theta^{\alpha\beta} = \frac{1}{4\pi}\left(g^{\alpha\mu}F_{\mu\lambda}F^{\lambda\beta} + \frac{1}{4}g^{\alpha\beta}F_{\mu\lambda}F^{\mu\lambda}\right) \quad (12.113)$$

Then $\partial_\alpha \Theta^{\alpha\beta} = \frac{-1}{c}F^{\beta\lambda}J_\lambda$ (12.118)

Extra point!

$\beta = 0,$ we have $\left(\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S}\right) = -\mathbf{J}\cdot\mathbf{E}$ (12.119') Conservation of energy

$\beta = 1, 2, 3,$ Conservation of momentum

11.3 Covariance of Electrodynamics (*continued*)

Consider the general form of the relativistic equation of motion in (47), $\frac{d}{dt}\mathbf{p}=\mathbf{F}$, where $\mathbf{F}$ is any force, such as the gravitational force.

Special case 1: $\mathbf{F}\parallel\mathbf{v}$ (one-dimensional problem)

$$F=\frac{d}{dt}(\gamma mv)=mv\underbrace{\frac{d\gamma}{dt}}+\gamma m\frac{dv}{dt}=\gamma m\frac{dv}{dt}\underbrace{\left(\gamma^2\frac{v^2}{c^2}+1\right)}=\gamma^3 m\frac{dv}{dt}$$

$$\boxed{\begin{aligned}\frac{d}{dt}\gamma&=\frac{d}{dt}\left(1-\frac{v^2}{c^2}\right)^{-1/2}\\&=\frac{-1}{2}\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left(\frac{-2v}{c^2}\right)\frac{dv}{dt}\quad=\gamma^3\frac{v}{c^2}\frac{dv}{dt}\end{aligned}}$$

$$\boxed{\begin{aligned}\gamma^2\frac{v^2}{c^2}+1&=\frac{v^2/c^2}{1-v^2/c^2}+1\\&=\frac{v^2/c^2+1-v^2/c^2}{1-v^2/c^2}=\frac{1}{1-v^2/c^2}=\gamma^2\end{aligned}}$$

$\Rightarrow F=\gamma^3 ma\Rightarrow$ Constant force does not cause constant acceleration.

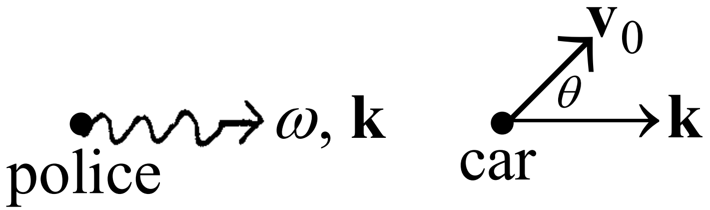
Special case 2: $\mathbf{F}\perp\mathbf{v}$ ($\Rightarrow\gamma=const.$, as in uniform circular motion)

$$\Rightarrow\mathbf{F}=\frac{d}{dt}\mathbf{p}=\frac{d}{dt}(\gamma m\mathbf{v})=\gamma m\frac{d}{dt}\mathbf{v}$$ (Undulator & Wiggler)

Questions: (i) It is sometimes said that a particle has two masses, $\gamma^3 m$ and γm. Why? (ii) The acceleration is not necessarily parallel to the force. Give an example. (iii) Relate (48) to (A.23).

11.3 Covariance of Electrodynamics (*continued*)

Problem 1: A police radar operates on a frequency of ω. What is the frequency received by the police after the signal is reflected from a car moving at the velocity $\mathbf{v}_0$?



shown in police frame

Solution: We do it in 2 steps.

Step 1. In the police frame, the radar sends a wave (ω, $\mathbf{k}$) toward the car, which is moving at velocity $\mathbf{v}_0$ (direction shown in the figure). Transforming ω to the car frame by (44), we obtain

$$\omega' = \gamma_0(\omega - v_0 k_z),$$

where k_z is the component of $\mathbf{k}$ along $\mathbf{v}_0$, i.e., $k_z = k\cos\theta$ (see figure.)

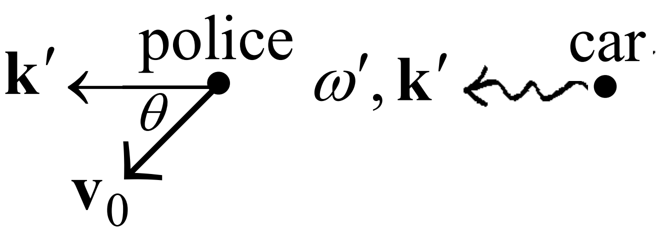
Thus,

$$\omega' = \gamma_0(\omega - k_z v_0) = \gamma_0\left(\omega - \overset{\boxed{k = \omega / c}}{k} v_0 \cos\theta\right) = \overset{\boxed{\gamma_0 = \left(1 - v_0^2 / c^2\right)^{-1/2}}}{\gamma_0}\omega\left(1 - \frac{v_0\cos\theta}{c}\right)$$

This is the Dopper-shifted frequency detected by the car. It is also the frequency of the wave reflected by the car as seen in the car frame.

Step 2. In the car frame (see figure), the car sends the reflected wave (ω', $\mathbf{k}'$) back to the car at the frequency

$$\omega' = \gamma_0 \omega \left(1 - \frac{v_0 \cos\theta}{c}\right)$$



shown in car frame

In the car frame, the police is moving at velocity $\mathbf{v}_0$ (direction shown in the fugure) relative to the car. Tranforming ω' to the police frame by (44), we obtain the frequency observed by the police (Doppler shifted again)

$$\omega'' = \gamma_0(\omega' - k'_z v_0) = \gamma_0(\omega' - k' v_0 \cos\theta) = \gamma_0 \omega' \left(1 - \frac{v_0 \cos\theta}{c}\right)$$

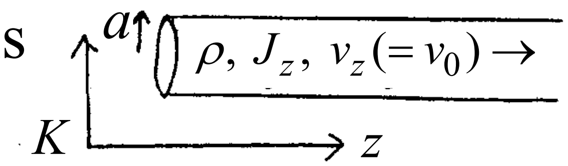
$$= \gamma_0^2 \omega \left(1 - \frac{v_0 \cos\theta}{c}\right)^2 \approx \omega \left(1 - 2\frac{v_0 \cos\theta}{c}\right) \quad \text{since } v_0 \ll c.$$

If the radar frequency is $f (= \omega / 2\pi) = 10^9$ Hz and the car moves away from the police ($\theta = 0$) at $v_0 = 150$ km/hr, the police would detect a frequency $f''(= \omega'' / 2\pi)$ shifted by $\Delta f \approx -f \dfrac{2v_0}{c} \approx -278$ Hz.

11.3 Covariance of Electrodynamics (*continued*)

Problem 2: An observer in the laboratory sees an infinite electron beam of radius a and uniform charge density ρ, moving axially at velocity v_0. What force does he see on an electron at a distance r ($\leq a$) from the axis? Assume the electron moves axially at the velocity v_0.

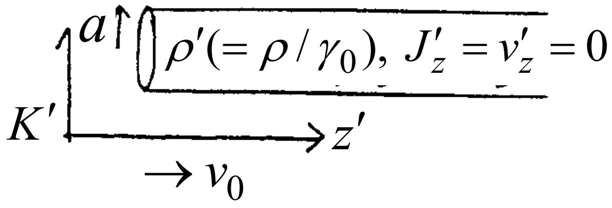
Solution: The problem can be readily solved in the lab frame. Here, we will take a long route for an exercise on some of the transformation equations just derived. $\left\{\nabla\cdot\mathbf{E}=4\pi\rho \ \ \& \ \ \nabla\times\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}=\frac{4\pi}{c}\mathbf{J}\right\}$



The current density J_z in the lab frame is

$$J_z=\rho v_0. \quad [\rho \text{ has a negative value.}]$$

By (37), we have, in the beam frame



$$J'_z=\gamma_0\left(J_z-v_0\rho\right)=0,$$

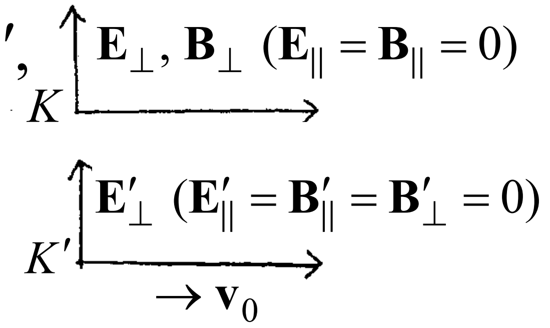
$$\rho'=\gamma_0\left(\rho-\frac{v_0}{c^2}J_z\right)=\gamma_0\rho\left(1-\frac{v_0^2}{c^2}\right)=\frac{\rho}{\gamma_0}.$$

We see that the lab frame ρ is greater than the beam frame ρ' by the factor γ_0. This is because every unit length of the beam in its rest frame is contracted by this factor when viewed in the lab frame.

11.3 Covariance of Electrodynamics (*continued*)

In the beam frame, $J'_z = 0$, $\rho' = \rho/\gamma_0$; hence, there is only a radial electric field. Gauss law, $\oint_{S'} \mathbf{E}' \cdot d\mathbf{a}' = 4\pi \int_{V'} \rho' d^3x'$, then gives $2\pi r' E'_r = 4\pi(\pi\rho' r'^2)$, for $r' \leq a$

$$\Rightarrow E'_r = 2\pi\rho' r' = \frac{2\pi\rho r}{\gamma_0}. \quad [r' = r,\ \rho' = \rho/\gamma_0]$$



We now transform $\mathbf{E}'_\perp \, (= E'_r \mathbf{e}_r)$ into lab-frame $\mathbf{E}_\perp$ and $\mathbf{B}_\perp$ by using the reverse transformation equations in (46), in which we set $\mathbf{v}_0 = v_0 \mathbf{e}_z$.

$$\begin{cases} \mathbf{E}_\perp = \gamma_0 \left(\mathbf{E}'_\perp - \frac{\mathbf{v}_0}{c} \times \cancel{\mathbf{B}'_\perp} \right) = \gamma_0 \mathbf{E}'_\perp = \gamma_0 \frac{2\pi\rho r}{\gamma_0} \mathbf{e}_r = 2\pi\rho r \mathbf{e}_r \\ \mathbf{B}_\perp = \gamma_0 \left(\cancel{\mathbf{B}'_\perp} + \frac{\mathbf{v}_0}{c} \times \mathbf{E}'_\perp \right) = \gamma_0 \left(\frac{v_0 \mathbf{e}_z}{c} \right) \times \frac{2\pi\rho r}{\gamma_0} \mathbf{e}_r = \frac{v_0}{c} 2\pi\rho r \mathbf{e}_\theta \end{cases}$$

Thus, the force $\mathbf{f}$ on an electron (in the lab frame) is

$$\mathbf{f} = -e\left(\mathbf{E} + \tfrac{1}{c} \mathbf{v} \times \mathbf{B} \right) = -e\left[2\pi\rho r \mathbf{e}_r + \tfrac{1}{c}(v_0 \mathbf{e}_z) \times \left(\tfrac{v_0}{c} 2\pi\rho r \mathbf{e}_\theta \right) \right]$$

$$= -2\pi e\rho r \left(1 - \frac{v_0^2}{c^2} \right) \mathbf{e}_r = -\frac{2\pi e\rho r}{\gamma_0^2} \mathbf{e}_r$$

[$e \equiv |e|$ is positive. For an electron beam, ρ is negative.]

Homework of Chap. 11 Problems: 3, 4, 5

Problem 11.3

Show explicitly that two successive Lorentz transformations in the same direction are equivalent to a single Lorentz transformation with a velocity

$$v = \frac{v_1 + v_2}{1 + \left(v_1 v_2 / c^2 \right)}$$

This is an alternative way to derive the parallel-velocity addition law.

Problem 11.4

A possible clock is shown in the figure. It consists of a flashtube F anda photocell P shielded so that each views only the mirror M, located a distance d away, and mounted rigidly with respect to the flashtube-photocell assembly. The electronic innards of the box are such that when the photocell responds to a light flash from the mirror, the flashtube is triggered with a negligible delay and emits a short flash toward the mirror. The clock thus "ticks" once every $(2d/c)$ seconds when at rest.

(a) Suppose that the clock moves with a uniform velocity v, perpendicular to the line from PF to M, relative to an observer. Using the second postulate of relativity, show by explicit geometrical or algebraic construction that the observer sees the relativistic time dilatation as the clock moves by.

(b) Suppose that the clock moves with a velocity v parallel to the line from PF to M. Verify that here, too, the clock is observed to tick more slowly, by the same time dilatation factor.

Problem 11.5

A coordinate system K' moves with a velocity $\vec{v}$ relative to another system K. In K' a particle has a velocity $\vec{u}'$ and an acceleration $\vec{a}'$. Find the Lorentz transformation law for accelerations, andshow that in the system K the components of acceleration parallel and perpendicular to $\vec{v}$ are

$$\vec{a}_{\parallel} = \frac{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}{\left(1 + \frac{\vec{v}\cdot\vec{u}'}{c^2}\right)} \vec{a}'_{\parallel} \qquad \vec{a}_{\perp} = \frac{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}{\left(1 + \frac{\vec{v}\cdot\vec{u}'}{c^2}\right)} \left(\vec{a}'_{\perp} + \frac{\vec{v}}{c^2} \times \left(\vec{a}' \times \vec{u}' \right) \right)$$

Homework of Chap. 11 Problems: 6, 9

Problem 11.6

Assume that a rocket ship leaves the earth in the year 2100. One of a set of twins born in 2080 remains on earth; the other rides in the rocket. The rocket ship is so constructed that it has an accelerationgin its own rest (frame this makes the occupants feel at home). It accelerates in a straight-line path for 5 years (by its own clocks), decelerates at the same rate for 5 more years, turns around, accelerates for 5 years, decelerates for 5 years, and lands on earth.

The twin in the rocket is 40 years old.

(a) What year is it on earth?

(b) How far away from the earth did the rocket ship travel?

Problem 11.9

An infinitesimal Lorentz transformation and its inverse can be written as

$$x'^{\alpha} = \left(g^{\alpha\beta} + \varepsilon^{\alpha\beta}\right) x_{\beta}$$

$$x^{\alpha} = \left(g^{\alpha\beta} + \varepsilon'^{\alpha\beta}\right) x'_{\beta}$$

where $\varepsilon^{\alpha\beta}$ and $\varepsilon'^{\alpha\beta}$ are infinitesimal.

(a) Show from the definition of the inverse that $\varepsilon'^{\alpha\beta} = -\varepsilon^{\alpha\beta}$

(b) Show from the preservation of the norm that $\varepsilon^{\beta\alpha} = -\varepsilon^{\alpha\beta}$

(c) By writing the transformation in terms of contravariant components on both sides of the equation, show that $\varepsilon^{\alpha\beta}$ is equivalent to the matrix L (11.93).

Homework of Chap. 11 Problems: 16, 19

Problem 11.16

In the rest frame of a conducting medium the current density satisfies Ohm's law $\mathbf{J}' = \sigma \mathbf{E}'$, where σ is the conductivity and primes denote quantities in the rest frame.

(a) Taking into account the possibility of convection current as well as conduction current, show that the covariant generalization of Ohm's law is

$$J^{\alpha} - \frac{1}{c^2}\left(U_{\beta} J^{\beta}\right) U^{\alpha} = \frac{\sigma}{c} F^{\alpha\beta} U_{\beta}$$

where U^{α} is the 4-velocity of the medium.

(b) Show that if the medium has a velocity $\mathbf{v} = c\boldsymbol{\beta}$ with respect to some inertial frame that the 3-vector current in that frame is

$$\mathbf{J} = \gamma\sigma\left[\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B} - \boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\right] + \rho\mathbf{v}$$

where ρ is the charge density observed in that frame.

Problem 11.19

A particle od mass M and 4-momentum P decays into two paricles of masses m_1 and m_2.

(a) Use the conservation of energy and momentum in the form, $p_2 = P - p_1$, and the invariance of scalar products of 4-vectors to show that the total energy of the first particle in the rest frame of the decaying particle is

$$E_1 = \frac{M^2 + m_1^{\ 2} + m_2^{\ 2}}{2M}$$

and E_2 is obtained by interchanging m_1 and m_2.

(b) Show that the kinetic energy T_i of the i-th particle in the same frames is

$$T_i = \Delta M\left(1 - \frac{m_i}{M} - \frac{\Delta M}{2M}\right)$$

Where $\Delta M = M - m_1 - m_2$ is the mass excess or Q value of the process.

(c) The charged pi-meson $(M = 139.6 \text{ MeV})$ decays into a mu-meson $(m_1 = 105.7 \text{ MeV})$ and a neutrino $(m_2 = 0 \text{ MeV})$. Calculate the kinetic energies of the mu-meson and the neutrino in the pi-meson's rest frame. The unique kinetic energy of the muon is the signature of a two-body decay. It entered importantly in the discovery of the pi-meson in photographic emulsions by Powell and coworkers in 1947.